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The Caustic, by Reflection, of a Circle.

BY T. J. I'A. BROMWICH.

Since the publication of Cayley's "Memoir upon Caustics,"[*] but little has been done in this direction. In the text-books of Geometrical Optics (e. g. Heath and Herman) Cayley's results and methods have been reproduced without alteration.

Cayley's work in the main is based upon the *point*-equation of the caustic (which is, of course, deduced from the equation of the reflected ray). But it is simpler, for most purposes, to work from the *line*-equation of the caustic,[†] or, what is really the same thing, to work with the coordinates of the reflected ray, expressed as algebraic polynomials of a parameter t.

Moreover, as in most questions dealing with circles and curves derived from circles, it is advantageous to use (instead of rectangular Cartesian coordinates ξ, η) the complex variables $x = \xi + i\eta$, $y = \xi - i\eta$. The use of x, y simplifies the arithmetic and makes most of the results more compact.[‡]

The following work is equivalent to that contained in Arts. VIII–XI XVII–XXV of Cayley's memoir. It appears that, for the problem of refraction (either at a line or at a circle) the method used below does not lead to any special simplification; and, accordingly, I have not reproduced any of my work on refracted caustics.

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[*] Phil. Trans. Roy. Soc. Lond., Vol. 147 (1857), p. 273 = Collected Math. Papers, Vol. 2, p. 336 (No. 145).

[†] That this is the more natural method seems clear from the singularities which are found; thus we have 6 cusps (at least 2 of which are real) and no inflexions.

[‡] For illustrations of this, see Morley's papers in the American Journal of Mathematics (Vols. XIII, XVI) and in Transactions of the American Mathematical Society (Vols. I, IV).

Incidentally, I have noted a few misprints in Cayley's memoir, which I have remarked in footnotes.

1. Take the reflecting circle as the unit circle

$$\xi^2 + \eta^2 = 1, \text{ or } xy = 1, \text{ if } x = \xi + i\eta, \; y = \xi - i\eta.$$

Then a point Q on the circle is given by the equations

$$x = t, \quad y = 1/t,$$

where

$$t = e^{i\phi},$$

and ϕ is the angle between the radius OQ and the axis of ξ.

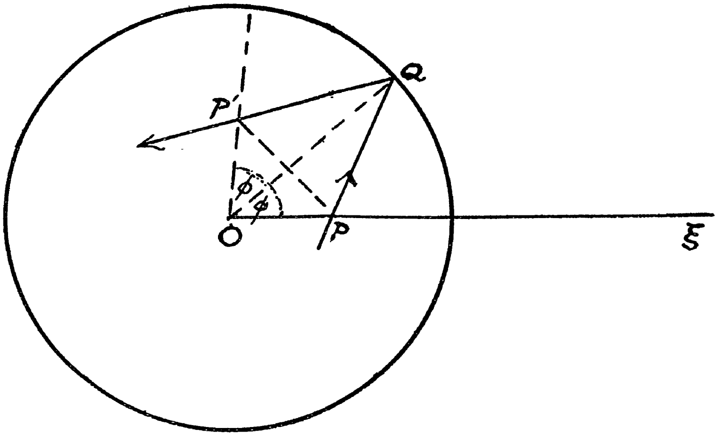
Suppose the rays start from a point P on the axis of ξ, say,

$$x = c, \quad y = c,$$

where, for convenience, c may be always supposed positive.

Then P', the reflection of P in OQ, is given by

$$OP' = OP = c,$$
$$P'OP = 2\phi,$$



so that, for P', we have

$$x = ct^2, \quad y = c/t^2.$$

The reflected ray, corresponding to an incident ray PQ, is clearly QP', whose equation is

$$\begin{vmatrix} x, & y, & 1 \\ t, & 1/t, & 1 \\ ct^2, & c/t^2, & 1 \end{vmatrix} = 0,$$

or $\qquad x(c-t)+yt^3(1-ct)+ct(t^2-1)=0,$

i. e., $\qquad -cyt^4+(c+y)\,t^3-(c+x)\,t+cx=0.$

$$\left.\begin{array}{l} x(c-t)+yt^3(1-ct)+ct(t^2-1)=0,\\ -cyt^4+(c+y)\,t^3-(c+x)\,t+cx\;=0.\end{array}\right\} \qquad (1)$$

Since the equation to the reflected ray is of degree 4 in t, the *class* of the caustic is also 4; and further, the caustic is unicursal. By differentiating the equation (1) with respect to t, and solving for x, y, we find

$$\frac{x}{ct^3(ct^3-3ct+2)}=\frac{y}{c(2t^3-3ct^2+c)}=\frac{1}{t^2\,[3ct^2-2(1+2c^2)\,t+3c]}\,,$$

so that the caustic is of degree 6. These expressions are easily seen to be equivalent to those given by Cayley in Art. XIX of his memoir (in which, however, the denominators should be $1-3a\cos\theta+2a^2$ instead of $1-3a\cos 2\theta+2a^2$). These expressions for x, y will not be of very much service to us; but it will be useful to obtain explicitly the point-equation of the caustic. The t discriminant of the quartic (1) is

$$g_2^3-27g_3^2=0,$$

where

$$g_2=\tfrac{1}{4}(x+c)(y+c)-xyc^2=\tfrac{1}{4}\,[xy(1-4c^2)+c(x+y)+c^2],$$
$$g_3=-\tfrac{1}{16}\,c\,(x-y)(xy-c^2).$$

hence the point-equation of the caustic is

$$-27c^2(x-y)^2(xy-c^2)^2+4\,[xy(1-4c^2)+c(x+y)+c^2]^3=0. \qquad (2)$$

This investigation of (2) is not essentially distinct from the second one given by Cayley (Art. XVIII).

2. Since the cubic is unicursal and of degree 6, it will have 10 nodes $(=\tfrac{1}{2}\,\overline{6-1}\,.\,\overline{6-2})$, including cusps; and since it is of class 4, it has 3 bitangents $(=\tfrac{1}{2}\,\overline{4-1}\,.\,\overline{4-2})$, including inflexional tangents. Thus, using Plücker's equations, we see that—

The caustic has 4 nodes, 6 cusps, 3 bitangents and no inflexions.

The equation given by Klein (Math. Ann., Vol. X) relating to the reality of singularities, shows that the number of *real* cusps is

$$2\,(1+\tau''-\delta''),$$

where τ'' is the number of isolated real bitangents, and δ'' is the number of isolated real nodes (acnodes).

3. *To determine the cusps*, we make the equation (1), regarded as an equation in t, have three equal roots α, α, α; the fourth root is then* $-\alpha$ (unless the repeated root is $t = 0$ or $t = \infty$, cases which are to be discussed separately). In this way we find the relations

$$\frac{1}{x} + \frac{1}{c} = \frac{2}{\alpha}, \quad \frac{1}{y} + \frac{1}{c} = 2\alpha, \quad \frac{x}{y} = \alpha^4,$$

leading to
$$\alpha^4 - 2\alpha^3 c + 2\alpha c - 1 = 0.$$

Thus we have
$$\text{(i).} \quad \alpha = \pm 1, \quad x = y = -c/(1 \pm 2c),$$

two cusps which are always *real;* and
$$\text{(ii).} \quad \alpha^2 - 2\alpha c + 1 = 0, \quad x = c\alpha^2, \quad y = c/\alpha^2,$$

two cusps which are real if $|\alpha| = 1$, or if $c < 1$; here the line joining the cusps is
$$\xi = \tfrac{1}{2}(x + y) = c\,(2c^2 - 1),$$

and they are on the circle $xy = c^2$, which passes through the bright point.

If $t = 0$ is a triple root of (1), we have
$$\frac{x}{0} = \frac{y}{1} = \frac{c}{0},$$

giving one of the circular points as a cusp; in like manner, if $t = \infty$ is a triple root, we find the other circular point as a cusp.

Hence we have now 6 cusps, agreeing with the number determined in §2; and, by examining equation (2), we verify the correctness of the work. The number of *real* cusps is 2 if $c > 1$, or 4 if $c < 1$. Thus,

$$\tau'' - \delta'' = 0, \quad c > 1,$$
$$\tau'' - \delta'' = 1, \quad c < 1.$$

4. *The bitangents.*

Construct first the relations between t_1, t_2, t_3, t_4 the parameters of the points of contact of the tangents drawn from any point to the caustic. From equation (1) we find
$$c\,(\Sigma t_1 + \Sigma t_1 t_2 t_3) = 1 + t_1 t_2 t_3 t_4, \quad \Sigma t_1 t_2 = 0.$$

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* This follows at once from the fact that equation (1) contains no terms in t^2.

Thus if $\quad p = t_1 + t_2, \quad q = t_1 t_2, \quad r = t_3 + t_4, \quad s = t_3 t_4,$

we have $\quad c(p + r + qr + ps) = 1 + qs, \quad q + s + pr = 0.$

From these equations r, s can, in general, be found in terms of p, q; but if t_1, t_2 belong to a bitangent, it is clear that t_3, t_4 (and so r, s) must be indeterminate. Thus for the bitangents, we have the conditions

$$(cp - 1) - q(cp - q) = 0, \quad c(1 + q) - p(cp - q) = 0.$$

The first of these conditions is

$$(1 - q)(cp - q - 1) = 0,$$

so we have two cases,

(i). $q = 1$, giving $cp^2 - p - 2c = 0$, i. e., $p = \frac{1}{2c}\left[1 \pm (1 + 8c^2)^{\frac{1}{2}}\right]$,

(ii). $cp = q + 1$, giving $p = 0$, $q = -1$.

Remembering that the bitangent is a tangent at $t = t_1$, $t = t_2$, we deduce that its equation must be

$$x + y\left[q - p^2 + cp\,(p^2 - 2q)\right] = c\,(p^2 - q - 1).$$

Thus, in case (i), the bitangent is $x + y = p$ or $\xi = \frac{1}{2}p$; that is,

$$\xi = \frac{1}{4c}\left[1 \pm (1 + 8c^2)^{\frac{1}{2}}\right],$$

while, in case (ii), the bitangent is $x - y = 0$ or $\eta = 0$. We have, accordingly, found the three bitangents; in case (i), the points of contact will be real if $|p| < 2$, which is always satisfied by the line $\xi = \frac{1}{4c}\left[1 - (1 + 8c^2)^{\frac{1}{2}}\right]$, but is only satisfied by the other bitangent if $c > 1$. In case (ii), the points of contact are always real, namely, the two cusps on the line $\eta = 0$.

Hence, from §§2, 3, we see that

$$c > 1, \quad \tau'' = 0, \quad \delta'' = 0,$$
$$c < 1, \quad \tau'' = 1, \quad \delta'' = 0.$$

The equations just found for the bitangents agree with those given by Cayley in Art. XXI of his memoir; Cayley finds, however, the tangents which are parallel to the axis of η, and these are clearly bitangents, owing to symmetry. From this point of view, we are to have equation (1) of the form

$$x + y = \text{const.},$$

so that

$$c - t - t^3 + ct^4 = 0,$$

which may be written in the form

$$c(t+1/t)^2-(t+1/t)-2c=0.$$

In this way we get back to the original forms.

So also we can find tangents parallel to the axis of ξ, from the condition

$$c-t+t^3-ct^4=0,$$

i. e.,

$$(1-t^2)[c(1+t^2)-t]=0.$$

The solution $t=\pm 1$ gives the third bitangent $\eta=0$; and the solution $c(t^2+1)=t$ gives the pair of parallel tangents

$$x-y=t-1/t=\pm i(4c^2-1)^{\frac{1}{2}}/c$$

or

$$2c\eta=\pm(4c^2-1)^{\frac{1}{2}},$$

as given by Cayley.

5. *The nodes.*

If (x, y) is a node of the caustic, the equation (1) is a perfect square, and so we can obtain the corresponding values of t (t_1, t_2, say) by writing $r=p$, $s=q$ in the first set of equations of §4. Thus

$$2cp(1+q)=1+q^2, \quad 2q+p^2=0,$$

and the corresponding values of x, y are given by

$$x=-q^2y, \quad y+c=2pcy.$$

Hence

$$x=qc(1+q)/(1-q), \quad y=c(q+1)/q(q-1),$$

and q is a root of the quartic

$$(1+q^2)^2+8c^2q(1+q)^2=0.$$

Thus there are four nodes, which are all imaginary; this can be proved by finding ξ, η. We have

$$\xi=\tfrac{1}{2}(x+y)=-\tfrac{1}{2}c(1+q)^2/q=-c(k+1)$$

if

$$k=\tfrac{1}{2}(q+1/q).$$

Now k satisfies

$$k^2+4c^2(k+1)=0,$$

and so

$$k=2c[-c\pm(c^2-1)^{\frac{1}{2}}],$$

thus

$$\xi=k^2/4c=c[-c\pm(c^2-1)^{\frac{1}{2}}]^2.$$

Also

$$\eta=(x-y)/2i=-ikc(1+q)/(1-q)$$

and

$$\left(\frac{1+q}{1-q}\right)^2=\frac{k+1}{k-1}=\frac{c\mp(c^2-1)^{\frac{1}{2}}}{3c\pm(c^2-1)^{\frac{1}{2}}},$$

since $$k = 2c/[-c \mp (c^2 - 1)^{\frac{1}{2}}].$$

These values of ξ, η agree with Cayley's, given in Art. XXII of his memoir; but it is possible to verify that the nodes are imaginary, without carrying out the calculations quite so far. In fact, if (x, y) is a real point, we must have

$$|x| = |y| \quad \text{or} \quad |q| = 1.$$

Thus k must be real and lie between $+1$ and -1; but it is clearly impossible to satisfy the equation

$$k^2 + 4c^2(k + 1) = 0$$

or real values of k, which are greater than -1. That is, (x, y) must be an imaginary point.

Since *all* the nodes have been proved to be imaginary, it follows that there are no real acnodes; which agrees with §4.

6. *The asymptotes.*

If we make two consecutive tangents parallel,[*] we find the condition

$$t^2[3c(t^2 + 1) - 2(1 + 2c^2)t] = 0$$

and rejecting the factor $t = 0$, which corresponds to the cuspidal tangent at one of the circular points, we have that the values of t giving the asymptotes are given by

$$3c(t^2 + 1) - 2(1 + 2c^2)t = 0,$$

or by $$3ct = 1 + 2c^2 \pm i(4c^2 - 1)^{\frac{1}{2}}(1 - c^2)^2.$$

The asymptotes are real if $|t| = 1$, or if $1 > c > \frac{1}{2}$; and then the explicit equations are given by substituting these values of t in equation (1). Thus, for instance, the perpendicular from the origin[†] on an asymptote is

$$\pm \tfrac{1}{2}\frac{ct(t^2 - 1)}{[(t^3 - ct^4)(c - t)]^{\frac{1}{2}}} = \pm \frac{\frac{1}{2}c(t - 1/t)}{[c(t + 1/t) - (1 + c^2)]^{\frac{1}{2}}} = [\tfrac{1}{3}(4c^2 - 1)]^{\frac{1}{2}}.$$

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[*] An alternative method is to make x, y infinite when expressed in terms of t. See the expression given in §1.

[†] The perpendicular from the origin on the line $lx + my + n = 0$ is $\pm \frac{1}{2} n/(lm)^{\frac{1}{2}}$.

The final equations of the asymptotes are

$$\eta = \pm \frac{(4c^2 - 1)^{\frac{3}{2}}}{(1 - c^2)^{\frac{1}{2}}(1 + 8c^2)}\left(\xi - \frac{3c}{4c^2 - 1}\right),$$

as found by Cayley (Art. XXI of his memoir).

7. *The caustic cuts the reflecting circle* in the points given by $xy = 1$, taken together with equation (2). Expressing everything in terms of $(x + y) = 2\xi$, we find

$$(c\xi - 1)^2\left[c\xi - 1 + \tfrac{9}{8}(1 - c^2)(1 + 3c^2)\right] = 0,$$

so that, if $c > 1$, the caustic touches the circle at the two real points given by $c\xi = 1$; if $c < 1$, this contact is imaginary.

The points given by $c\xi = 1 - \frac{9}{8}(1 - c^2)(1 + 3c^2)$ are imaginary if $c > 1$; for we have*

$$c^2(\xi^2 - 1) = \tfrac{1}{64}(1 - c^2)(1 - 9c^2)^3;$$

and so these points are real only if $1 > c > \frac{1}{3}$. Summing up, we see that

$c > 1$, the caustic has double contact with the circle in real points ($\xi = 1/c$).
$1 > c > \frac{1}{3}$, the caustic cuts the circle in two real points,

$$c\xi = 1 - \tfrac{9}{8}(1 - c^2)(1 + 3c^2).$$

$\frac{1}{3} > c$, the caustic is within the circle.

8. *The foci of the caustic.*

If the tangent, as given by equation (1), passes through the circular point

$$\frac{x}{0} = \frac{y}{1} = \frac{1}{0}$$

we must have

$$t^3(1 - ct) = 0.$$

The value $t = 0$ gives the tangent $x = 0$, which is the cuspidal tangent at this circular point; and $t = 1/c$ gives the tangent $x = 1/c$.

Hence the four tangents from this circular-point are included in the equation

$$x^3(cx - 1) = 0.$$

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* Cayley's arithmetic appears to be a little wrong here. For he states that $\xi = \pm 1$ gives $(c \pm 1)(27c^2 + 9c + 1) = 0$, and he deduces the condition for reality $c < 1$, without including $c > \frac{1}{3}$ (Art. XXI). But later on (Art. XXV) he states the correct result for $c < \frac{1}{3}$.

Similarly, the four tangents from the other circular point are determined by

$$y^3(cy-1)=0.$$

It follows that $x=0$, $y=0$ is a triple focus, and that the only other real focus is $x=1/c$, $y=1/c$.

To make up the total of 16 foci belonging to curve of class 4, it should be observed that the triple focus counts for 9; and that there are two more imaginary foci ($x=1/c$, $y=0$ and $x=0$, $y=1/c$) each counting for 3; then $9+6+1=16$

Hence:

The real foci of the caustic consist of (i) *a triple focus at the centre of the reflecting circle; and* (ii), *a single focus at the point inverse to the bright point with respect to the reflecting circle.*

9. We have now all the *general* theorems for the caustic; but the special cases, $c=1$, ∞, deserve a little consideration. Take first the case $c=1$, when *the origin of light is on the reflecting circle.* The equation (1) becomes

$$(t-1)[x+yt^3-t(t+1)]=0$$

and equation (2) becomes

$$(x+y-2)^2\,[27x^2y^2-18xy-4(x+y)-1]=0.$$

Thus the curve degenerates into the line $x+y-2$ counted twice and the cardioid

$$27x^2y^2-18xy-4(x+y)-1=0,$$

as given by Cayley in Art. XXIII of his memoir. Starting from the tangential equation

$$x+yt^3-t(t+1)=0$$

we find that points on the caustic are given by*

$$3x=2t+t^2,\quad 3y=2/t+1/t^2,$$

an epicycloid (cardioid) given by the rolling of a circle of radius $\frac{1}{3}$ on an equal circle. Similarly, the line equation gives, for the degenerate form, the point $x=y=1$ and the same cardioid.

In this case, the class is 3 and the degree is 4; there is one bitangent, $\xi=-\frac{1}{2}$; there are 3 cusps (at $\xi=-\frac{1}{3}$, $\eta=0$ and at the two circular points);

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* These values are also found by putting $c=1$ in the expressions given in §1, for x, y in terms of t.

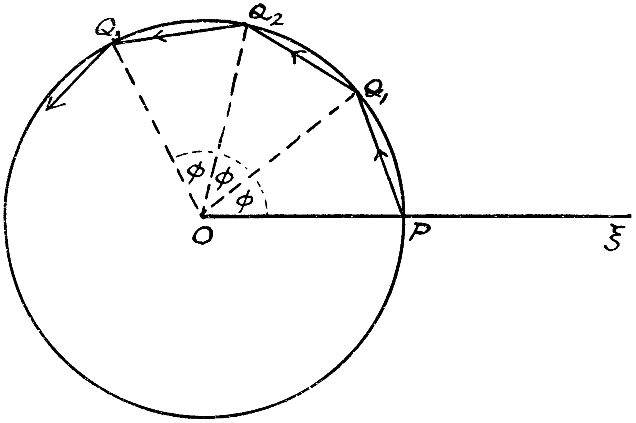
the imaginary nodes disappear;* there are no real asymptotes and there are no inflexions (as in the general caustic). The point of contact of the cardioid and line $\xi = 1$ counts as the other 3 cusps of the general case; while $\xi = 1$ is the limiting form of a bitangent.

If $c = \infty$, so that *the incident rays are parallel to* $o\xi$, we find that equation (1) gives

$$x - yt^4 + t(t^2 - 1) = 0$$

leading to
$$4x = 3t - t^3, \quad 4y = 3/t - 1/t^3$$

for points on the caustic. This proves that the caustic is an epicycloid of class 4 and of degree 6. This curve has the two real cusps $\xi = \pm \frac{1}{2}$, $\eta = 0$; and the two imaginary nodes $\xi = 0$, $\eta = \pm 2i/\sqrt{3}$, the remaining singularities are all at the two circular points, there being two cusps and a node at each, as is easily verified by taking the penultimate form. The three bitangents are $\eta = 0$, $\xi = \pm 1/\sqrt{2}$.



This special caustic can be obtained by rolling a circle of radius $\frac{1}{4}$ upon a circle of radius $\frac{1}{2}$. The point equation is

$$4(4xy - 1)^3 + 27(x - y)^2 = 0,$$

as given by Cayley (Art. XXIV).

10. In the two cases mentioned in §9, the caustic can be easily found after

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* If the cardioid and line are taken together, the 4 nodes coincide in pairs at the two imaginary intersections of the line and the cardioid. That is, at the points

$$\xi = 1, \quad \eta = \pm 2i/\sqrt{3}.$$

n reflections. Taking first the case when the origin of light is on the reflecting circle, it is easily seen that the n^{th} point of incidence Q_n is

$$x = t^n, \quad y = 1/t^n.$$

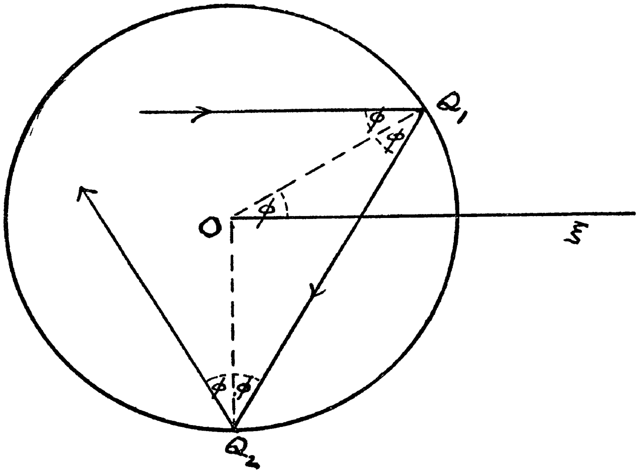
Hence the equations to the ray, after n reflections, is

$$x + yt^{2n+1} = t^n (t + 1),$$

since this is the equation to the line $Q_n\, Q_{n+1}$. If we write $n = 1$, we get back to the equation of §8. We deduce at once, by differentiating with respect to t, the coordinates of a point on the envelope in the form

$$(2n + 1)\, x = (n + 1)\, t^n \;+ nt^{n+1},$$
$$(2n + 1)\, y = (n + 1)/t^n + n/t^{n+1}.$$

These equations show that the envelope is an epicycloid of class $(2n + 1)$ and of degree $2(n + 1)$. It may be found geometrically by rolling a circle of radius $n/(2n + 1)$ upon one of radius $1/(2n + 1)$; agreeing with what was found for the case $n = 1$.



The determination of the singularities need not be undertaken here, as those of the epicycloids are well known.

For the case when the incident rays are parallel to ξ, we see that Q_{n+1} is the point

$$x = (-1)^n\, t^{2n+1}, \quad y = (-1)^n/t^{2n+1}.$$

Thus the n^{th} reflected ray (since it joins Q_{n+1} to Q_n) is

$$x - yt^{4n} = (-1)^n\, t^{2n-1}\, (t^2 - 1)$$

leading to the caustic

$$4nx = (-1)^n\,[(2n-1)\,t^{2n+1} - (2n+1)\,t^{2n-1}],$$
$$4ny = (-1)^n\,[(2n-1)/t^{2n+1} - (2n+1)/t^{2n-1}].$$

This is an epicycloid of class $4n$ and of degree $(4n+2)$. It may be obtained by rolling a circle of radius $(2n-1)/4n$ upon one of radius $1/2n$. In both these cases the epicycloids have a finite number of *real* nodes.

These cases are given by Cayley (Arts. X, XI).

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